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First, keep in mind that whenever you see a map $F : M \to L$ between manifolds, you can always *locally* regard M and L as open subsets of some Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^r$, respectively, by choosing local coordinates around the relevant points. Once you do that, the derivative (pushforward) D_pF is just the usual Jacobian matrix of the coordinate representative of F.

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1. Why you do *not* need an elaborate choice of charts

In many textbook or classroom treatments, one emphasizes that for a map $F : M \to L$, you must pick:

1. A coordinate chart $\big(U, \varphi_U\big)$ on M with $p \in U \subset M$ and $\varphi_U(U) \subset \mathbb{R}^n$.
2. A coordinate chart $\big(V, \psi_V\big)$ on L with $F(p) \in V \subset L$ and $\psi_V(V) \subset \mathbb{R}^r$.

Then, the "coordinate representative" of F near p is

$$\widehat{F} \;=\; \psi_V \circ F \circ \varphi_U^{-1} \;:\; \varphi_U(U) \;\longrightarrow\; \psi_V(V)\,.$$

To compute the derivative (as an $r \times n$ matrix) at the point $\varphi_U(p) \in \mathbb{R}^n$, you just take the usual partial derivatives of the components of $\widehat{F}$.

However, **if** your manifolds M and L are *already* sitting inside Euclidean spaces in a nice way (for instance $M_{2\times 2} \cong \mathbb{R}^4$ and $S^1 \subset \mathbb{R}^2$), you can usually take the most obvious "identity" charts:

- On $M_{2\times 2}$, use the identity map $\varphi_U = \mathrm{Id}\colon (a, b, c, d) \mapsto (a, b, c, d)$ from $\mathbb{R}^4$ to itself.
- On S^1, at least *locally* around $(\cos\theta, \sin\theta)$, you can either use a standard parametrization by θ (if you really want a 1-dimensional chart) *or* you can temporarily view $S^1 \subset \mathbb{R}^2$ and just compute the derivative in $\mathbb{R}^2$.

Because the derivative of a smooth map between manifolds can be computed by *any* choice of coordinate charts, it is usually *simplest* to choose the obvious "identity" charts in the ambient Euclidean spaces.

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2. Concrete example: $F(A) = \big(\cos(\det A), \sin(\det A)\big)$

Let

$$A \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \in M_{2\times 2} \cong \mathbb{R}^4.$$

Then

$$\det(A) \;=\; ad - bc.$$

The map $F\colon M_{2\times 2} \to S^1 \subset \mathbb{R}^2$ is given by

$$F(a,b,c,d) \;=\; \big(\cos(ad-bc),\ \sin(ad-bc)\big).$$

If you are content to view $S^1 \subset \mathbb{R}^2$, then from the perspective of $\mathbb{R}^4 \to \mathbb{R}^2$, the derivative $D_{(a,b,c,d)}F$ is simply the 2×4 Jacobian matrix of the function

$$\big(\cos(\det(A)),\, \sin(\det(A))\big)$$

with respect to (a,b,c,d).

Step 1: Compute partial derivatives of $\det(A)$

$$\det(A) \;=\; ad - bc,$$

so

$$\frac{\partial}{\partial a}(\det(A)) = d, \quad \frac{\partial}{\partial b}(\det(A)) = -c, \quad \frac{\partial}{\partial c}(\det(A)) = -b, \quad \frac{\partial}{\partial d}(\det(A)) = a.$$

Step 2: Chain rule for $\cos(\det(A))$ and $\sin(\det(A))$

$$\frac{\partial}{\partial a}\big(\cos(\det(A))\big) \;=\; -\sin(\det(A)) \cdot \frac{\partial}{\partial a}(\det(A)) \;=\; -\sin(ad-bc)\cdot d,$$

and similarly for partials w.r.t. b, c, d. Likewise for the $\sin(\det(A))$ component.

Putting it all together, the Jacobian matrix

$$D_{(a,b,c,d)}F \;=\; \begin{pmatrix} \frac{\partial}{\partial a}\cos(\det(A)) & \frac{\partial}{\partial b}\cos(\det(A)) & \cdots & \frac{\partial}{\partial d}\cos(\det(A)) \\ \frac{\partial}{\partial a}\sin(\det(A)) & \frac{\partial}{\partial b}\sin(\det(A)) & \cdots & \frac{\partial}{\partial d}\sin(\det(A)) \end{pmatrix}.$$

One can factor out the common factor $\pm\sin(\det(A))$ or $\pm\cos(\det(A))$, but explicitly you get

$$D_{(a,b,c,d)}F \;=\; \begin{pmatrix} -\sin(ad-bc)\,d & -\sin(ad-bc)\,(-c) & -\sin(ad-bc)\,(-b) & -\sin(ad-bc)\,a \\ \cos(ad-bc)\,d & \cos(ad-bc)\,(-c) & \cos(ad-bc)\,(-b) & \cos(ad-bc)\,a \end{pmatrix}.$$

That is a valid "matrix representative" of the derivative in the usual sense of a map $\mathbb{R}^4 \to \mathbb{R}^2$. If you prefer to regard it purely as a map $T_pM \to T_{F(p)}S^1$, then you note that $T_{F(p)}S^1$ is a 1-dimensional subspace in $\mathbb{R}^2$; however, the usual approach is precisely to compute in $\mathbb{R}^2$ and then interpret the result as a map into that tangent line.

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3. "Choosing an arbitrary value for the coordinate chart"

Often, the phrase "choosing a coordinate chart" sounds more complicated than it really is. In practical computations:

1. **On $M_{2\times 2}$:** We identify

$$M_{2\times 2} \;\cong\; \mathbb{R}^4, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\longleftrightarrow\; (a, b, c, d).$$

   That *is* our local coordinate chart around any point p. There is nothing more to do.

2. **On S^1:** We know $S^1 \subset \mathbb{R}^2$. Strictly, a 1D manifold chart around a point $(x_0, y_0) \in S^1$ can be taken, e.g., by solving $y = \sqrt{1 - x^2}$ locally or by using $\theta \mapsto (\cos\theta, \sin\theta)$. But for the sake of computing the derivative, you can just as well treat S^1 as embedded in $\mathbb{R}^2$ and compute the 2D Jacobian. Locally, that identifies $T_{(x_0,y_0)}S^1$ with the corresponding line in $\mathbb{R}^2$.

Hence, you never really need to do anything like "choose a fancy parameterization" unless you specifically want a 1-dimensional coordinate on S^1. If your goal is just to find the matrix of partial derivatives, using the identity chart on $\mathbb{R}^4$ and the ambient coordinates in $\mathbb{R}^2$ is totally fine.

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4. Conclusion

- **Is it necessary to pick some arbitrary coordinate chart?**
  In practice, *no*: for a map $F : \mathbb{R}^4 \to S^1 \subset \mathbb{R}^2$, you can (and almost always do) just use the identity coordinates on $\mathbb{R}^4$ and the ambient coordinates on $\mathbb{R}^2$.

- **Why is that enough?**
  Because from the viewpoint of differential geometry, *any* smooth chart around p in $M_{2\times 2}$ and around $F(p)$ in S^1 gives the *same* derivative up to isomorphism. The identity chart on $\mathbb{R}^4$ is the simplest possible. Similarly, embedding S^1 in $\mathbb{R}^2$ is the simplest way to see its tangent space and compute partials.

So if you are "confused" by the statement "choose an arbitrary coordinate chart," you can simply interpret it as: "Use the standard (identity) coordinates on $\mathbb{R}^4$ for $M_{2\times 2}$ and the standard embedding of $S^1 \subset \mathbb{R}^2$." That is a perfectly valid choice, and it leads directly to the usual partial-derivative (Jacobian) computation.

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